Technical Notes

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Traction Boundary Conditions in Hybrid-Stress Finite-Element Model

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Introduction

THE so-called hybrid-stress finite-element model (belonging to a general class of hybrid models based on modified variational principles with relaxed interelement continuity requirements, as reviewed in Ref. 1) has been developed extensively by Pian,[2] Tong and Pian,[3] and others. The variational basis of the hybrid-stress model, which is based on a modified complementary energy principle, can be stated thus: Let the stress field σ_{ij} in each finite element V_m $(m=1\cdots N)$ be assumed such that the conditions‡ 1) $\sigma_{ij,j}+\bar{F}_i=0$ ($\bar{F}_i$ are body forces) and 2) $\sigma_{ij}n_j\equiv T_i=\bar{T}_i$ on $S_{\sigma m}$ (where $S_{\sigma m}$ is the portion of the boundary ∂V_m of V_m upon which tractions are prescribed) are met a priori. However, the interelement traction reciprocity condition§ $T_i^+=-T_i^-$ at ρ_m (ρ_m is that portion of ∂V_m which is an interelement interface) need not be satisfied a priori. It then can be shown[1] that the stationary condition of the functional

$$\pi_{HS}(\sigma_{ij},\tilde{u}_{ip})=_m\left(\int_{V_m}B(\sigma_{ij})\mathrm{d}v-\int_{S_{u_m}}T_i\bar{u}_i\mathrm{d}s-\int_{\rho_m}T_i\tilde{u}_{ip}\mathrm{d}s\right)\tag{1}$$

(where B is the complementary energy density, and $\tilde{u}_{ip}$ are compatible interelement boundary displacements) leads to the Euler equations: 1) kinematic compatibility condition in V_m; 2) displacement boundary conditions at S_{u_m} (that portion of ∂V_m where displacement are prescribed); 3) the condition $u_i^+=\tilde{u}_{ip}=u_i^-$ at ρ_m; and 4) $T_i^++T_i^-=0$ at ρ_m. Thus, $\tilde{u}_{ip}$ play the role of Lagrange multipliers to enforce condition 4. It is noted that the traction boundary condition on $S_{\sigma m}$ is an "essential boundary condition" in the formulation based on Eq. (1). For instance, the exact satisfaction, a priori, of traction boundary conditions on the surface of a cutout or a crack has been found to be essential for obtaining accurate results for stress-concentration or stress-intensity factors, using the hybrid-stress formulation in Refs. 4 and 5. The a priori enforcement of $\sigma_{ij}n_j=\bar{T}_i$ on $S_{\sigma m}$ may be possible when the segments of $S_{\sigma m}$ are straight and parallel to the coordinate axes ($n_j=1$ or 0) used to define the geometry of the element. However, when $S_{\sigma m}$ is an arbitrary curve, or even not simply parellel to the coordinate axes used, it is, in general, impossible to choose σ_{ij} in V_m such that $\sigma_{ij}n_j=\bar{T}_i$ on $S_{\sigma m}$ is satisfied a priori. In such a case, Ref. 6 suggests choosing additional Lagrange multipliers u_i on $S_{\sigma m}$ to be of the same form as that of $\tilde{u}_{ip}$ on ρ_m, and furthermore setting $\tilde{u}_{ip}\equiv\bar{u}_i$ on S_{u_m}, in order that the functional in Eq. (1) may be modified as

$$\pi_{HS}(\sigma_{ij},\tilde{u}_{ip})=\sum_m\left(\int_{V_m}B(\sigma_{ij})\mathrm{d}v-\int_{\partial V_m}T_i\tilde{u}_{ip}\mathrm{d}s+\int_{S_{\sigma_m}}\bar{T}_i\tilde{u}_{ip}\mathrm{d}s\right)\tag{2}$$

the stationary condition of which, subject to the constraint $\delta\sigma_{ij,j}=0$, is

$$\delta\pi_{HS}=\sum_m\left\{\int_{V_m}\left[\frac{\partial B}{\partial\sigma_{ij}}-\frac{1}{2}(f_{ij}+f_{j,i})\right]\delta\sigma_{ij}+\int_{\partial V_m}\delta T_i(f_i-\tilde{u}_{ip})\mathrm{d}s-\int_{\rho_m}\delta\tilde{u}_{ip}T_i\mathrm{d}\rho-\int_{S_{\sigma_m}}(T_i-\bar{T}_i)\delta\tilde{u}_{ip}\mathrm{d}s\right\}=0\tag{3}$$

where f_i are differentable functions in V_m. For the present purpose, it is instructive to the study the last integral in Eq. (3) from the point of view of a weighted residual equation, when the appropriate finite-element assumptions.¶

$$\underset{\sim}{\sigma}=\underset{\approx}{P}\underset{\sim}{\beta};\quad \tilde{u}_{ip}=\underset{\approx}{L}\underset{\sim}{q}\tag{4}$$

are made, as in the usual application.[6] In Eq. (4), $\underset{\sim}{\beta}$ are determined parameters, $\underset{\sim}{\sigma}$ satisfies equilibrium in V_m, and $\underset{\approx}{L}$ are polynomials defined at ∂V_m which interpolate for boundary displacements uniquely in terms of nodal displacements $\underset{\sim}{q}$. Representing the boundary tractions corresponding to assumed $\underset{\sim}{\sigma}$, as $\underset{\sim}{T}=\underset{\approx}{R}\underset{\sim}{\beta}$, one can see that the last integral in Eq. (3) can be written as

$$\int_{S_{\sigma_m}}(T_i-\bar{T}_i)\delta\tilde{u}_{ip}\mathrm{d}s=\int_{S_{\sigma_m}}[\underset{\sim}{\beta}^T\underset{\approx}{R}^T-\bar{\underset{\sim}{T}}^T]\underset{\approx}{L}\delta\underset{\sim}{q}\tag{5}$$

It is seen from Eq. (5) that, for vanishing of the preceding integral for arbitrary $\delta\underset{\sim}{q}$, the integral average of the residual error in traction boundary conditions, weighted with respect to each of the functions of $\underset{\approx}{L}$ at $S_{\sigma m}$, is set to zero. The number of weighted residual equations for traction boundary conditions, as seen from Eq. (5), is the same as the number of admissible $\delta\underset{\sim}{q}$ at $S_{\sigma m}$; however, this number, in general, is considerably less than the number of assumed traction modes (number of β's) in a typical finite-element formulation.[2,4,6] From the general theory of weighted residual methods,[7] this implies that the satisfaction of traction boundary conditions through Eq. (5) will not be numerically accurate; however, this accuracy can be improved with the increase of weighting functions in $\underset{\approx}{L}$ in Eq. (5), which in turn implies that the number of nodal degrees of freedom $\underset{\sim}{q}$ must be increased at $S_{\sigma m}$. On the contrary, from the viewpoint of computational efficiency, it is desirable to keep the number of nodal $\underset{\sim}{q}$'s the same for all elements and as small as possible. In Refs. 4 and 5, results obtained for crack problems wherein traction

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‡The usual Cartesian tensor notation is used; a comma followed by an index such as i denotes partial differentiation with respect to X_i.

§The superscripts plus and minus denote the left- and right-hand-side vicinities of ρ_m, respectively.

¶(x) denotes a matrix, r a column vector, and superscript T a transpose.

boundary conditions were satisfied through Eq. (5), with no additional $\mathbf{q}$'s at S_{σ_m}, were found to be highly inaccurate. The object of this Note is to present simple methods whereby the satisfaction of traction boundary conditions, a posteriori, through the variational principle, can be highly accurate even for arbitrary curved S_{σ_m}, without increasing the number of $\mathbf{q}$'s at S_{σ_m} as discussed previously.

Methods to Improve Accuracy of Traction Boundary Conditions

Instead of the formulation of Ref. 6 based on Eq. (2), the constraint condition $T_i = \bar{T}$ on S_{σ_m} is introduced directly into Eq. (11), using as independent set of Lagrange multipliers $\tilde{u}_{i\sigma}$ thus:

$$\pi_{HS}(\sigma_{ij};\tilde{u}_{ip};\tilde{u}_{i\sigma}) = \sum_m \left(\int_{V_m} B(\sigma_{ij})\,\mathrm{d}v - \int_{S_{u_m}} T_i \bar{u}_i \mathrm{d}s - \int_{\rho_m} T_i \tilde{u}_{ip}\mathrm{d}\rho - \int_{S_{\sigma_m}} (T_i - \bar{T}_i)\tilde{u}_{i\sigma}\mathrm{d}s \right) \tag{6}$$

whose first variation is

$$\delta\pi_{HS} = \sum_m \left\{ \int_{V_m} \left[\frac{\partial B}{\partial \sigma_{ij}} - \frac{1}{2}(f_{i,j}+f_{j,i}) \right] \delta\sigma_{ij} - \int_{S_{u_m}} \delta T_i(\bar{u}_i - f_i)\mathrm{d}s - \int_{\rho_m} \delta T_i(\tilde{u}_{ip} - f_i) - \int_{\rho_m} T_i \delta\tilde{u}_{ip} - \int_{S_{\sigma_m}} \delta T_i(\tilde{u}_{i\sigma} - f_i) - \int_{S_{\sigma_m}} (T_i - \bar{T}_i)\delta\tilde{u}_{i\sigma} \right\} \tag{7}$$

Thus, for each finite element, one assumes

$$\boldsymbol{\sigma} = \mathbf{P}\boldsymbol{\beta} \text{ in } V_m \tag{8a}$$

$$\tilde{u}_{ip} = \mathbf{L}\mathbf{q} \text{ at } \rho_m \tag{8b}$$

$$\tilde{u}_{i\sigma} = \mathbf{B}\boldsymbol{\alpha} \text{ at } S_{\sigma_m} \tag{8c}$$

where $\mathbf{P}$, $\boldsymbol{\beta}$, $\mathbf{L}$, and $\mathbf{q}$ are as defined in Eq. (4), $\mathbf{B}$ are arbitrary order polynomials in the boundary coordinates at $S_{\sigma m}$, and $\boldsymbol{\alpha}$ are undetermined parameters. By increasing the number of functions in $\mathbf{B}$, it can be seen from Eq. (7) that the error residual in traction boundary conditions can be orthogonalized with respect to as many functions as desired, and thus the accuracy of traction boundary conditions can be as high as desired. Using Eq. (8) in Eq. (7), one obtains

$$\pi_{HS} = \sum_m \left\{ \frac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} - \boldsymbol{\beta}^{T}\mathbf{U} - \boldsymbol{\beta}^T\mathbf{G}\mathbf{q} - \boldsymbol{\beta}^T\mathbf{C}^T\boldsymbol{\alpha} + \mathbf{Q}^T\boldsymbol{\alpha} \right\} \tag{9}$$

where

$$\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} = 2\int_{V_m} B(\boldsymbol{\sigma})\mathrm{d}V; \quad \bar{\mathbf{U}} = \int_{S_{u_m}} \mathbf{R}^T\bar{\mathbf{u}}\mathrm{d}s \tag{10a}$$

$$\mathbf{G} = \int_{\rho_m} \mathbf{R}^T\mathbf{L}\mathrm{d}\rho; \quad \bar{\mathbf{Q}} = \int_{S_{\sigma_m}} \mathbf{B}^T\bar{\mathbf{T}} \tag{10b}$$

$$\mathbf{C} = \int_{S_{\sigma_m}} \mathbf{B}^T\mathbf{R}\mathrm{d}s \tag{10c}$$

Varying π_{HS} with respect to the independent element variables $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$, respectively, one obtaines for each element

$$\mathbf{H}\boldsymbol{\beta} - \mathbf{U} - \mathbf{G}\mathbf{q} - \mathbf{C}^T\boldsymbol{\alpha} = 0 \tag{11a}$$

$$\mathbf{C}\boldsymbol{\beta} - \mathbf{Q} = 0 \tag{11b}$$

Equation (11b) is the algebraic expression of the traction boundary condition constraint condition on the chosen stress field. The number of algebraic equations represented by Eq. (11b) is the same as the number of α's chosen at S_{σ_m}. Denoting by a, b, c, and d the number of $\boldsymbol{\beta}$'s, number of $\boldsymbol{\alpha}$'s, number of independent element $\mathbf{q}$'s, and the number of possible rigid body modes, respectively, for the element the argument of Ref. 6 leads to the following condition for solvability of Eq. (11):

$$(a-b) \geq (c-d) \tag{12}$$

Since the number a of $\boldsymbol{\beta}$'s can be arbitrary, the preceding inequality is trivial to satisfy in any element formulation. Solving for $\boldsymbol{\beta}$'s and $\boldsymbol{\alpha}$ from Eqs. (11) in terms of $\mathbf{q}$, and substituting in Eq. (9), the element stiffness matrix $\mathbf{k}_m$ can be obtained as

$$\mathbf{k}_m = \mathbf{G}^T[\mathbf{H}^{-1}\mathbf{C}^T(\mathbf{C}^T\mathbf{H}^{-1}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{H}^{-1}]\mathbf{G} \tag{13}$$

Alternatively, instead of seeking to set the weighted residual error in traction boundary conditions to zero as done previously one may use the familiar concept of boundary-point matching technique in *identically* satisfying traction boundary conditions at an arbitrary number of points at S_{σ_m}. In this alternative method, the constraint conditions on the chosen stress field become

$$T_i^p = \bar{T}_i^p \text{ at } p=1,2,\ldots P \text{ points at } S_{\sigma_m} \tag{14}$$

Using the notation $\mathbf{T}^P$ to represent the master vector of traction components at all P points, the preceding condition can be written as

$$\mathbf{T}^P = \bar{\mathbf{T}}^P \tag{15}$$

The foregoing pointwise constraint is introduced into the variational formulation of Eq. (1) through the additional term

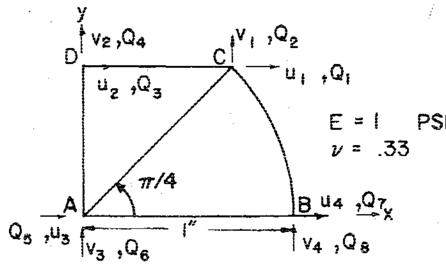
$$(\mathbf{T}^P - \bar{\mathbf{T}}^P)^T\boldsymbol{\lambda}_p \tag{16}$$

where $\boldsymbol{\lambda}_p$ is a vector of pointwise Lagrange multipliers. Comparing Eq. (16) and the last term in Eq. (6), it is seen that $\boldsymbol{\lambda}_p$ can be interpreted as the integral averages of $\tilde{u}_{i\sigma}$ weighted with as yet unknown functions. Expressing $\mathbf{T}^P$ in terms of the assumed stress field $\boldsymbol{\sigma}$, the additional term of Eq. (16) is written as

$$(\mathbf{C}^P\boldsymbol{\beta} - \bar{\mathbf{T}}^P)^T\boldsymbol{\lambda}_p \tag{17}$$

where $\mathbf{C}^P$ is obtained simply by substituting the coordinates of points $p=1,2,\ldots P$ on S_{σ_m} into the functions $\mathbf{P}$ of Eq. (8). It easily can be seen, by comparing the terms in Eq. (17) and the last two terms in the functional of Eq. (9), that the expression for the element stiffness matrix in the present point-matching technique is identical to that in Eq. (13), except that the $\mathbf{C}$ in Eq. (13) should be replaced by $\mathbf{C}^P$ of Eq. (17). If the total number of traction boundary condition constraint conditions at P points is denoted by b, it is necessary, as before, that $(a-b) \geq c-d$, with a, c, and d defined as earlier. This point-matching technique has the obvious advantage over the earlier method in that no additional numerical integrations are needed at S_{σ_m}, and, by increasing the number of matching points, the accuracy can be improved to the desired level.

In this connection, it may be noted that Wolf[8] has presented a method to improve the traction boundary conditions in the hybrid-stress model. His approach now can be interpreted as a special case of the presently described point-matching technique wherein the number of points chosen was the same as the "nodes" at $S_{\sigma m}$ where displacement degrees of freedom are to be solved for thus lacking the generality of the present approach, where the number of matching points is



NODAL FORCE	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
BASED ON REF. 6	-.167	-0.800	-0.035	-0.417	0.002	0.767	0.120	0.451
INDEPENDENT $\tilde{u}_{i\sigma}$	0.120	-0.148	-0.052	-0.415	-0.164	0.410	0.095	0.153
BOUND-COLLO.	0.120	-0.148	-0.052	-0.415	-0.164	0.410	0.095	0.153

Fig. 1 Element geometry.

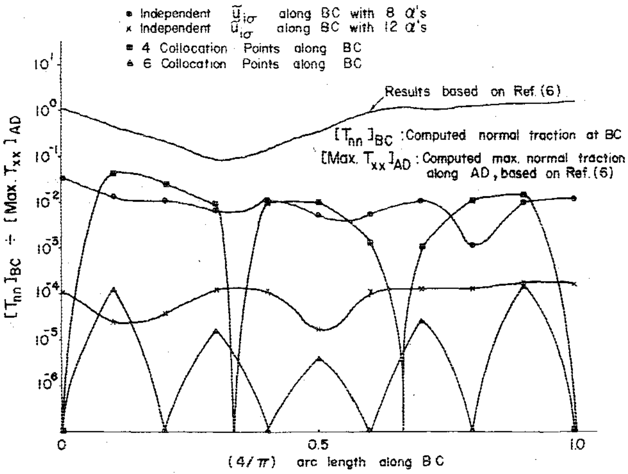


Fig. 2 Residual normal traction along BC.

arbitrary and independent of the "nodes" of the finite element at $S_{\sigma m}$. Moreover, it appears that in Ref. 8 the pointwise constraints were added to the functional in Eq. (2), thus making the formulation in Ref. 8 somewhat inconsistent.

Numerical Results

To test the efficiency of the developed procedures, a simple problem of a single plane stress finite-element $ABCD$ with eight degrees of freedom $\underset{\sim}{q}$, as shown in Fig. 1, is devised. Side BC, which is an arc of a circle, is assumed to be traction-free. The solution of the problem for prescribed displacements $u_3 = u_1 = v_1 = u_2 = v_2 = 1$ $v_3 = v_4 = u_4 = 0$ is studied. In all of the cases studied, an equilibrated element-stress field is assumed, as derived from the Airy stress function, which is a polynomial with 18 parameters $\underset{\sim}{\beta}$, consisting of complete quadratic to quintic terms in (x,y). In the formulation based on Pian and Tong,[6] using Eq. (2), the boundary displacement $\tilde{u}_{ip}$ all along $ABCD$ is assumed to be linear along each segment and is interpolated uniquely in terms of the nodal displacements of the respective segment. For the method involving an independently assumed $\tilde{u}_{i\sigma}$ along BC, each displacement component was assumed to be a simple polynomial in the arc length along BC, starting from a quadratic to a quintic; thus the number of α's ranged from 6 to 12. The matching points for the boudary-collection method were taken to be alternatively four and six, respectively, the points being equidistant along BC. In both of the methods, inequality (12) clearly is satisfied. Moreover, in both of the methods, $\tilde{u}_{ip}$ along AB, CD, and DA were assumed to be identical to that in the formulation based on Ref. 6, described previously.

The stiffness matrices obtained using the two present methods were identical to the fourth significant digit but differed significantly from that obtained through the formulation of Pian and Tong.[6] This is reflected in the general nodal forces, as shown in Fig. 1. The computed normal tractions along BC, normalized with respect to the maximum normal traction along line AD as computed through the formualtion of Ref. 6, are shown in Fig. 2. The residual tractions along BC using the formulation of Ref. 6 are seen to be of the same order as the maximum traction along AD; whereas, under both of the present formulations, they are at least an order of magnitude smaller when eight α's or four matching points are used, and, as can be seen from Fig. 2, these residual tractions are several orders of magntiude smaller when the number of α's or the number of collocation points is increased. Results similar to those in Fig. 2 were obtained for tangential tractions also but are not shown here for lack of space. Using the point-matching method for traction boundary conditions in conjunction with the hybrid-stress model, excellent results for problems of bending of plates with stress-free boundary conditions are reported in Ref. 5.

Conclusion

Two simple, efficient methods are presented to enforce, as accurately as desired, the traction boundary conditions along arbitrarily curved boundaries using the hybrid-stress finite-element model.

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Rod and Beam Finite Element Matrices and Their Accuracy

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EXACT dynamic stiffness matrices for rods and beams have been given by Henshell and Warburton,[1] Akesson,[2] and others. Reference 2 includes an extensive bibliography and gives an historical perspective to this method of structural

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